GFD I - Midterm Exam

Handed out 2/6/2012, Due Monday 2/15/2012 at the start of class

You may consult your class notes, Vallis, Holton, Gill, and Kundu & Cohen, but you should not discuss this with your classmates. Problem 1 is worth 30 points, and 2 is worth 20.

1) This is a variation on the Rossby adjustment problem that was done in class in lecture 3.4. Imagine that you are on an infinite *f*-plane, with a flat bottom at $z = -H$, and with the free surface at $z = \eta$. Initially we will assume that the surface is flat, but that there is a disturbance in the velocity field with the mathematical form

$$v = -v_0 \sin(kx)$$

where v_0 is some positive constant with units of velocity. The velocity is independent of depth initially (and for all time). Also, there is no E-W velocity initially, so $u = 0$. Throughout this problem you may assume that the linearized Shallow Water Equations apply:

X-MOM $\quad u_t - fv = -g\eta_x$

Y-MOM $\quad v_t + fu = -g\eta_y$

MASS $\quad \eta_t + H(u_x + v_y) = 0$

(a) Is the initial condition in geostrophic balance?
(b) What terms may we drop from the equations based on the nature of the initial condition?
(c) Taking the curl of the momentum equations (Y-MOM$_x$ – X-MOM$_y$) we came up with a relation between the vorticity, $\zeta = v_x - u_y$, and the surface height. Most importantly, we were able to relate these two fields at any time to their initial values. What is the equation giving this relation in this case?
(d) The flow will evolve, generating some Poincare waves which will radiate away (this is not formally true if the actual disturbance is infinite in extent, but would be true if it only existed over some limited spatial extent, for example as a region of limited *x*-extent

but many wavelengths across, analogous to a "wave packet"). Assuming that the flow attains a steady state that is geostrophic, what is the equation for this final state in terms of η?

(e) Solve the result of (d) to find the solution for the eventual steady surface height field.

(f) Also find the eventual steady velocity field.

[HINT: make your expressions for (e-f) as algebraically simple as possible, making use of the expression $(ka)^2$, where $a = (gH)^{1/2} f^{-1}$ is the Rossby radius of deformation.]

(g) Explain why your final expression makes sense in terms of vortex stretching.

(h) Which direction did fluid parcels near the *x*-origin have to move in order to bring about the changes from the initial to final states? Choose a place that has non-zero motion, and state which position you chose. Figure this out using a time integral of Y-MOM (note that $\eta_y = 0$ for all time). How *far* did they move, expressed as a dimensionless number times the Rossby radius?

(i) The kinetic and potential energies of shallow water flow, per unit horizontal area, are given by:

$$KE_A = \frac{1}{2}\rho H\left(u^2 + v^2\right)$$

$$PE_A = \frac{1}{2}\rho g \eta^2$$

What is KE_A / PE_A for the final state? [Express your answer in terms of ka. Note that the horizontal average of the cosine or sine squared is ½.]

(j) What is the ratio of total energy, $E_A = KE_A + PE_A$, for the final versus initial states? How much of the energy remains if $ka = 1$? How much of the energy remains if $1/k >> a$ (disturbance wavelength much greater than the Rossby radius)?

2) Consider forced, damped flow on the *f*-plane. We will assume that the flow is initially at rest, and that the forcing and damping are infinite in horizontal extent. This means that the resulting velocities have no spatial structure, and are only functions of time. The forcing and damping will be assumed to be independent of depth, e.g. like a wind stress at the top, averaged over the thickness of the flow. The governing equations are:

X-MOM $\quad u_t - fv = -Ru$

Y-MOM $\quad v_t + fu = F - Rv$

where F is the forcing (units of acceleration), which we will assume to be "turned on" to a constant value at $t = 0$. The damping is a "Rayleigh friction" with timescale $1/R$. Note that the surface is flat at the start, and remains so for all time.

(a) What will the velocity be like in the final steady state? Draw a sketch of the force balance, and compare it to the force balance that would occur for flow with $R = 0$.

(b) Solve for the full time evolution of the velocity. [HINT: mathematically this is simplified by expressing the two equations as a single equation in the complex number $U \equiv u + iv$.]

(c) Sketch the trajectory of a fluid parcel which is at the origin at $t = 0$.